

IDENTITIES INVOLVING FROBENIUS-EULER POLYNOMIALS ARISING FROM NON-LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we consider non-linear differential equations which are closely related to the generating functions of Frobenius-Euler polynomials. From our non-linear differential equations, we derive some new identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order.

1. INTRODUCTION

Let $u \in \mathbb{C}$ with $u \neq 1$. Then the Frobenius-Euler polynomials are defined by generating function as follows:

$$(1) \quad F_u(t, x) = \frac{1-u}{e^t-u} e^{xt} = \sum_{n=0}^{\infty} H_n(x | u) \frac{t^n}{n!}, \quad (\text{see [1,2]}).$$

In the special case, $x = 0$, $H_n(0 | u) = H_n(u)$ are called the n -th Frobenius-Euler numbers (see [2]).

By (1), we get

$$(2) \quad H_n(x | u) = \sum_{l=0}^n \binom{n}{l} x^{n-l} H_l(u) \quad \text{for } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$

Thus, by (1) and (2), we get the recurrence relation for $H_n(u)$ as follows:

$$(3) \quad H_0(u) = 1, \quad (H(u) + 1)^n - H_n(u) = \begin{cases} 1-u & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

with the usual convention about replacing $H(u)^n$ by $H_n(u)$ (see [2,10,12]).

The Bernoulli and Euler polynomials can be defined by

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

In the special case, $x = 0$, $B_n(0) = B_n$ are the n -th Bernoulli numbers and $E_n(0) = E_n$ are the n -th Euler numbers.

The formula for a product of two Bernoulli polynomials are given by

$$(4) \quad B_m(x) B_n(x) = \sum_{r=0}^{\infty} \left(\binom{m}{2r} n + \binom{n}{2r} m \right) \frac{B_{2r} B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1} \frac{m! n!}{(m+n)!} B_{m+n},$$

where $m+n \geq 2$ and $\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{m(m-1)\cdots(m-n+1)}{n!}$ (see [1,3]).

From (1), we note that $H_n(x \mid -1) = E_n(x)$. In [10], Nielson also obtained similar formulas for $E_n(x)E_m(x)$ and $E_m(x)B_n(x)$.

In view point of (4), Carlitz have considered the following identities for the Frobenius-Euler polynomials as follows:

(5)

$$H_m(x \mid \alpha)H_n(x \mid \beta) = H_{m+n}(x \mid \alpha\beta) \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \\ + \frac{\alpha(1-\beta)}{1-\alpha\beta} \sum_{r=0}^m \binom{m}{r} H_r(\alpha) H_{m+n-r}(x \mid \alpha\beta) + \frac{\beta(1-\alpha)}{1-\alpha\beta} \sum_{s=0}^n \binom{n}{s} H_s(\beta) H_{m+n-s}(x \mid \alpha\beta),$$

where $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 1$, $\beta \neq 1$ and $\alpha\beta \neq 1$ (see [2]).

In particular, if $\alpha \neq 1$ and $\alpha\beta = 1$, then

$$H_m(x \mid \alpha)H_n(x \mid \alpha^{-1}) = -(1-\alpha) \sum_{r=1}^m \binom{m}{r} H_r(\alpha) \frac{B_{m+n-r+1}(x)}{m+n-r+1} \\ - (1-\alpha^{-1}) \sum_{s=1}^n \binom{n}{s} H_s(\alpha^{-1}) \frac{B_{m+n-s+1}(x)}{m+n-s+1} \\ + (-1)^{n+1} \frac{m!n!}{(m+n+1)!} (1-\alpha) H_{m+n+1}(\alpha).$$

For $r \in \mathbb{N}$, the n -th Frobenius-Euler polynomials of order r are defined by generating function as follows:

$$F_u^r(t, x) = \underbrace{F_u(t, x) \times F_u(t, x) \times \cdots \times F_u(t, x)}_{r \text{ - times}} \\ = \underbrace{\left(\frac{1-u}{e^t - u} \right) \times \left(\frac{1-u}{e^t - u} \right) \times \cdots \times \left(\frac{1-u}{e^t - u} \right)}_{r \text{ - times}} e^{xt} \\ = \sum_{n=0}^{\infty} H_n^{(r)}(x \mid u) \frac{t^n}{n!} \quad \text{for } u \in \mathbb{C} \text{ with } u \neq 1. \quad (6)$$

In the special case, $x = 0$, $H_n^{(r)}(0 \mid u) = H_n^{(r)}(u)$ are called the n -th Frobenius-Euler numbers of order r (see [1-15]).

In this paper we derive non-linear differential equations from (1) and we study the solutions of non-linear differential equations. Finally, we give some new and interesting identities and formulae for the Frobenius-Euler polynomials of higher order by using our non-linear differential equations.

2. COMPUTATION OF SUMS OF THE PRODUCTS OF FROBENIUS-EULER NUMBERS AND POLYNOMIALS

In this section we assume that

$$(7) \quad F = F(t) = \frac{1}{e^t - u}, \quad \text{and} \quad F^N(t, x) = \underbrace{F \times \cdots \times F}_{N \text{ - times}} e^{xt} \quad \text{for } N \in \mathbb{N}.$$

Thus, by (7), we get

$$(8) \quad F^{(1)} = \frac{dF(t)}{dt} = \frac{-e^t}{(e^t - u)^2} = -\frac{1}{e^t - u} + \frac{u}{(e^t - u)^2} = -F + uF^2.$$

By (8), we get

$$(9) \quad \begin{aligned} F^{(1)}(t, x) &= F^{(1)}(t)e^{tx} = -F(t, x) + uF^2(t, x), \\ \text{and} \quad F^{(1)} + F &= uF^2. \end{aligned}$$

Let us consider the derivative of (8) with respect to t as follows:

$$(10) \quad 2uFF' = F'' + F'.$$

Thus, by (10) and (8), we get

$$(11) \quad 2!u^2F^3 - 2uF^2 = F'' + F'.$$

From (11), we note that

$$(12) \quad 2!u^2F^3 = F^{(2)} + 3F' + 2F, \quad \text{where } F^{(2)} = \frac{d^2F}{dt^2}.$$

Thus, by the derivative of (12) with respect to t , we get

$$(13) \quad 2!u^23F^2F' = F^{(3)} + 3F^{(2)} + 2F^{(1)}, \quad \text{and} \quad F^{(1)} = uF^2 - F.$$

By (13), we see that

$$(14) \quad 3!u^3F^4F' = F^{(3)} + 6F^{(2)} + 11F^{(1)} + 6F.$$

Thus, from (14), we have

$$3!u^4F^4(t, x) = F^{(3)}(t, x) + 6F^{(2)}(t, x) + 11F^{(1)}(t, x) + 6F(t, x).$$

Continuing this process, we set

$$(15) \quad (N-1)!u^{N-1}F^N = \sum_{k=0}^{N-1} a_k(N)F^{(k)},$$

where $F^{(k)} = \frac{d^kF}{dt^k}$ and $N \in \mathbb{N}$.

Now we try to find the coefficient $a_k(N)$ in (15). From the derivative of (15) with respect to t , we have

$$(16) \quad N!u^{N-1}F^{N-1}F^{(1)} = \sum_{k=0}^{N-1} a_k(N)F^{(k+1)} = \sum_{k=1}^N a_{k-1}(N)F^{(k)}.$$

By (8), we easily get

$$(17) \quad N!u^{N-1}F^{N-1}F^{(1)} = N!u^{N-1}F^{N-1}(uF^2 - F) = N!u^N F^{N+1} - N!u^{N-1}F^N.$$

From (16) and (17), we can derive the following equation (18):

$$(18) \quad \begin{aligned} N!u^N F^{N+1} &= N(N-1)!u^{N-1}F^N + \sum_{k=1}^N a_{k-1}(N)F^{(k)} \\ &= N \sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^N a_{k-1}(N)F^{(k)}. \end{aligned}$$

In (15), replacing N by $N+1$, we have

$$(19) \quad N!u^N F^{N+1} = \sum_{k=0}^N a_k(N+1)F^{(k)}.$$

By (18) and (19), we get

$$\begin{aligned}
 (20) \quad \sum_{k=0}^N a_k(N+1)F^{(k)} &= N!u^N F^{N+1} \\
 &= N \sum_{k=0}^{N-1} a_k(N)F^{(k)} + \sum_{k=1}^N a_{k-1}(N)F^{(k)}.
 \end{aligned}$$

By comparing coefficients on the both sides of (20), we obtain the following equations:

$$(21) \quad Na_0(N) = a_0(N+1), \quad a_N(N+1) = a_{N-1}(N).$$

For $1 \leq k \leq n-1$, we have

$$(22) \quad a_k(N+1) = Na_k(N) + a_{k-1}(N),$$

where $a_k(N) = 0$ for $k \geq N$ or $k < 0$. From (21), we note that

$$(23) \quad a_0(N+1) = Na_0(N) = N(N-1)a_0(N-1) = \cdots = N(N-1) \cdots 2a_0(2).$$

By (8) and (15), we get

$$(24) \quad F + F' = uF^2 = \sum_{k=0}^1 a_k(2)F^{(k)} = a_0(2)F + a_1(2)F^{(1)}.$$

By comparing coefficients on the both sides of (24), we get

$$(25) \quad a_0(2) = 1, \quad \text{and} \quad a_1(2) = 1.$$

From (23) and (25), we have $a_0(N) = (N-1)!$. By the second term of (21), we see that

$$(26) \quad a_N(N+1) = a_{N-1}(N) = a_{N-2}(N-1) = \cdots = a_1(2) = 1.$$

Finally, we derive the value of $a_k(N)$ in (15) from (22).

Let us consider the following two variable function with variables s, t :

$$(27) \quad g(t, s) = \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^N}{N!} s^k, \quad \text{where } |t| < 1.$$

By (22) and (27), we get

$$\begin{aligned}
 (28) \quad & \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_{k+1}(N+1) \frac{t^N}{N!} s^k \\
 &= \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} Na_{k+1}(N+1) \frac{t^N}{N!} s^k + \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^N}{N!} s^k \\
 &= \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} Na_{k+1}(N) \frac{t^N}{N!} s^k + g(t, s).
 \end{aligned}$$

It is not difficult to show that

$$\begin{aligned}
 (29) \quad & \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} N a_{k+1}(N) \frac{t^N}{N!} s^k = \frac{1}{s} \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} N a_{k+1}(N) \frac{t^N}{N!} s^{k+1} \\
 &= \frac{1}{s} \sum_{N \geq 1} \sum_{1 \leq k \leq N} a_k(N) \frac{t^N}{(N-1)!} s^k = \frac{1}{s} \sum_{N \geq 1} \left(\sum_{0 \leq k \leq N} a_k(N) \frac{t^N s^k}{(N-1)!} - \frac{a_0(N) t^N}{(N-1)!} \right) \\
 &= \frac{1}{s} \sum_{N \geq 1} \left(\sum_{0 \leq k \leq N} a_k(N) \frac{t^N}{(N-1)!} s^k - t^N \right) = \frac{t}{s} \left(\sum_{N \geq 1} \sum_{0 \leq k \leq N} a_k(N) \frac{t^{N-1} s^k}{(N-1)!} - \frac{1}{1-t} \right) \\
 &= \frac{t}{s} \left(g'(t, s) - \frac{1}{1-t} \right).
 \end{aligned}$$

From (28) and (29), we can derive the following equation:

$$(30) \quad \sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_{k+1}(N+1) \frac{t^N s^k}{N!} = \frac{t}{s} \left(g'(t, s) - \frac{1}{1-t} \right) + g(t, s).$$

(31)

$$\begin{aligned}
 & \text{The left hand side of (13)} = \sum_{N \geq 2} \sum_{1 \leq k \leq N-2} a_{k+1}(N) \frac{t^{N-1}}{(N-1)!} s^k \\
 &= \sum_{N \geq 2} \sum_{1 \leq k \leq N-1} a_k(N) \frac{t^{N-1} s^{k-1}}{(N-1)!} = \frac{1}{s} \left(\sum_{N \geq 2} \sum_{1 \leq k \leq N-1} a_k(N) \frac{t^{N-1}}{(N-1)!} s^k \right) \\
 &= \frac{1}{s} \left(\sum_{N \geq 2} \left(\sum_{0 \leq k \leq N-1} a_k(N) \frac{t^{N-1}}{(N-1)!} s^k - a_0(N) \frac{t^{N-1}}{(N-1)!} \right) \right) \\
 &= \frac{1}{s} \left(\sum_{N \geq 2} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^{N-1}}{(N-1)!} s^k - \frac{t}{1-t} \right) \\
 &= \frac{1}{s} \left(\sum_{N \geq 1} \sum_{0 \leq k \leq N-1} a_k(N) \frac{t^{N-1}}{(N-1)!} s^k - a_0(1) - \frac{t}{1-t} \right) = \frac{1}{s} \left(g'(t, s) - \frac{1}{1-t} \right).
 \end{aligned}$$

By (30) and (31), we get

$$(32) \quad g(t, s) + \frac{t}{s} \left(g'(t, s) - \frac{1}{1-t} \right) = \frac{1}{s} \left(g'(t, s) - \frac{1}{1-t} \right).$$

Thus, by (32), we easily see that

$$(33) \quad 0 = g(t, s) + \frac{t-1}{s} g'(t, s) + \frac{1-t}{s(1-t)} = g(t, s) + \frac{t-1}{s} g'(t, s) + \frac{1}{s}.$$

By (33), we get

$$(34) \quad g(t, s) + \frac{t-1}{s} g'(t, s) = -\frac{1}{s}.$$

To solve (34), we consider the solution of the following homogeneous differential equation:

$$(35) \quad 0 = g(t, s) + \frac{t-1}{s} g'(t, s).$$

Thus, by (35), we get

$$(36) \quad -g(t, s) = \frac{t-1}{s} g'(t, s).$$

By (33), we get

$$(37) \quad \frac{g'(t, s)}{g(t, s)} = \frac{s}{1-t}.$$

From (37), we have the following equation:

$$(38) \quad \log g(t, s) = -s \log(1-t) + C.$$

By (38), we see that

$$(39) \quad g(t, s) = e^{-s \log(1-t)} \lambda \quad \text{where } \lambda = e^C.$$

By using the variant of constant, we set

$$(40) \quad \lambda = \lambda(t, s).$$

From (39) and (40), we note that

$$(41) \quad \begin{aligned} g'(t, s) &= \frac{dg(t, s)}{dt} = \lambda'(t, s) e^{-s \log(1-t)} + \frac{\lambda(t, s) e^{-s \log(1-t)}}{1-t} s \\ &= \lambda'(t, s) e^{-s \log(1-t)} + \frac{g(t, s)}{1-t} s, \end{aligned}$$

where $\lambda'(t, s) = \frac{d\lambda(t, s)}{dt}$.

By multiply $\frac{t-1}{s}$ on both sides in (41), we get

$$(42) \quad \frac{t-1}{s} g'(t, s) + g(t, s) = \lambda' \frac{t-1}{s} e^{-s \log(1-t)}.$$

From (34) and (42), we get

$$(43) \quad -\frac{1}{s} = \lambda' \frac{t-1}{s} e^{-s \log(1-t)}.$$

Thus, by (43), we get

$$(44) \quad \lambda' = \lambda'(t, s) = (1-t)^{s-1}.$$

If we take indefinite integral on both sides of (44), we get

$$(45) \quad \lambda = \int \lambda' dt = \int (1-t)^{s-1} dt = -\frac{1}{s} (1-t)^s + C_1,$$

where C_1 is constant.

By (39) and (45), we easily see that

$$(46) \quad g(t, s) = e^{-s \log(1-t)} \left(-\frac{1}{s} (1-t)^s + C_1 \right).$$

Let us take $t = 0$ in (46). Then, by (27) and (46), we get

$$(47) \quad 0 = -\frac{1}{s} + C_1 \quad , \quad C_1 = \frac{1}{s}.$$

Thus, by (46) and (47), we have

$$(48) \quad \begin{aligned} g(t, s) &= e^{-s \log(1-t)} \left(\frac{1}{s} - \frac{1}{s} (1-t)^s \right) = \frac{1}{s} (1-t)^{-s} (1 - (1-t)^s) \\ &= \frac{(1-t)^{-s} - 1}{s} = \frac{1}{s} (e^{-s \log(1-t)} - 1). \end{aligned}$$

From (48) and Taylor expansion, we can derive the following equation (49):

$$(49) \quad \begin{aligned} g(t, s) &= \frac{1}{s} \sum_{n \geq 1} \frac{s^n}{n!} (-\log(1-t))^n = \sum_{n \geq 1} \frac{s^{n-1}}{n!} \left(\sum_{l=1}^{\infty} \frac{t^l}{l} \right)^n \\ &= \sum_{n \geq 1} \frac{s^{n-1}}{n!} \left(\sum_{l_1=1}^{\infty} \frac{t^{l_1}}{l_1} \times \cdots \times \sum_{l_n=1}^{\infty} \frac{t^{l_n}}{l_n} \right) \\ &= \sum_{n \geq 1} \frac{s^{n-1}}{n!} \sum_{N \geq n} \left(\sum_{l_1 + \cdots + l_n = N} \frac{1}{l_1 l_2 \cdots l_n} \right) t^N. \end{aligned}$$

Thus, by (49), we get

$$(50) \quad \begin{aligned} g(t, s) &= \sum_{k \geq 0} \frac{s^k}{(k+1)!} \sum_{N \geq k+1} \left(\sum_{l_1 + \cdots + l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} \right) t^N \\ &= \sum_{N \geq 1} \left(\sum_{0 \leq k \leq N-1} \frac{N!}{(k+1)!} \sum_{l_1 + \cdots + l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} \right) \frac{t^N}{N!} s^k. \end{aligned}$$

From (27) and (50), we can derive the following equation (51):

$$(51) \quad a_k(N) = \frac{N!}{(k+1)!} \sum_{l_1 + \cdots + l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}}.$$

Therefore, by (15) and (51), we obtain the following theorem.

Theorem 1. For $u \in \mathbb{C}$ with $u \neq 1$, and $N \in \mathbb{N}$, let us consider the following non-linear differential equation with respect to t :

$$(52) \quad F^N(t) = \frac{N}{u^{N-1}} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1 + \cdots + l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} F^{(k)}(t),$$

where $F^{(k)}(t) = \frac{d^k F(t)}{dt^k}$ and $F^N(t) = \underbrace{F(t) \times \cdots \times F(t)}_{N \text{ - times}}$. Then $F(t) = \frac{1}{e^t - u}$ is a solution of (52).

Let us define $F^{(k)}(t, x) = F^{(k)}(t) e^{tx}$. Then we obtain the following corollary.

Corollary 2. For $N \in \mathbb{N}$, we set

$$(53) \quad F^N(t, x) = \frac{N}{u^{N-1}} \sum_{k=0}^N \frac{1}{(k+1)!} \sum_{l_1 + \cdots + l_{k+1} = N} \frac{1}{l_1 l_2 \cdots l_{k+1}} F^{(k)}(t, x).$$

Then $\frac{e^{tx}}{e^t - u}$ is a solution of (53).

From (1) and (6), we note that

$$\begin{aligned}
 \frac{1-u}{e^t-u} &= \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \\
 \text{and} \\
 \underbrace{\left(\frac{1-u}{e^t-u}\right) \times \left(\frac{1-u}{e^t-u}\right) \times \cdots \times \left(\frac{1-u}{e^t-u}\right)}_{N-\text{ times}} &= \sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!},
 \end{aligned}
 \tag{54}$$

where $H_n^{(N)}(u)$ are called the n -th Frobenius-Euler numbers of order N .

By (7) and (54), we get

$$\begin{aligned}
 F^N(t) &= \underbrace{\left(\frac{1}{e^t-u}\right) \times \left(\frac{1}{e^t-u}\right) \times \cdots \times \left(\frac{1}{e^t-u}\right)}_{N-\text{ times}} \\
 &= \frac{1}{(1-u)^N} \underbrace{\left(\frac{1-u}{e^t-u}\right) \times \left(\frac{1-u}{e^t-u}\right) \times \cdots \times \left(\frac{1-u}{e^t-u}\right)}_{N-\text{ times}} \\
 &= \frac{1}{(1-u)^N} \sum_{l=0}^{\infty} H_l^{(N)}(u) \frac{t^l}{l!},
 \end{aligned}
 \tag{55}$$

and

$$F(t) = \left(\frac{1-u}{e^t-u}\right) \left(\frac{1}{1-u}\right) = \frac{1}{1-u} \sum_{l=0}^{\infty} H_l(u) \frac{t^l}{l!}.$$

From (55), we note that

$$F^{(k)}(t) = \frac{d^k F(t)}{dt^k} = \sum_{l=0}^{\infty} H_{l+k}(u) \frac{t^l}{l!}.$$

Therefore, by (52), (55) and (56), we obtain the following theorem.

Theorem 3. For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have

$$H_n^{(N)}(u) = N \left(\frac{1-u}{u}\right)^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{H_{n+k}(u)}{l_1 l_2 \cdots l_{k+1}}.$$

From (55), we can derive the following equation:

$$\begin{aligned}
 \sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!} &= \underbrace{\left(\frac{1-u}{e^t-u} \right) \times \left(\frac{1-u}{e^t-u} \right) \times \cdots \times \left(\frac{1-u}{e^t-u} \right)}_{N \text{ - times}} \\
 &= \left(\sum_{l_1=0}^{\infty} H_{l_1}(u) \frac{t_{l_1}^1}{l_1!} \right) \times \cdots \times \left(\sum_{l_N=0}^{\infty} H_{l_N}(u) \frac{t_{l_N}^N}{l_N!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \frac{H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u) n!}{l_1! l_2! \cdots l_N!} \right) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \dots, l_N} H_{l_1}(u) \cdots H_{l_N}(u) \right) \frac{t^n}{n!}.
 \end{aligned}
 \tag{57}$$

Therefore, by (57), we obtain the following corollary.

Corollary 4. For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}
 \sum_{l_1+\cdots+l_N=n} \binom{n}{l_1, \dots, l_N} H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u) \\
 = N \left(\frac{1-u}{u} \right)^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{H_{n+k}(u)}{l_1! l_2! \cdots l_{k+1}!}.
 \end{aligned}$$

By (53), we obtain the following corollary.

Corollary 5. For $N \in \mathbb{N}$, $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}
 H_n^{(N)}(x|u) \\
 = N \left(\frac{1-u}{u} \right)^{N-1} \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\cdots+l_{k+1}=N} \frac{1}{l_1! l_2! \cdots l_{k+1}!} \sum_{m=0}^n \binom{n}{m} H_{m+k}(u) x^{n-m}.
 \end{aligned}$$

From (6), we note that

$$\begin{aligned}
 \sum_{n=0}^{\infty} H_n^{(N)}(x|u) \frac{t^n}{n!} &= \underbrace{\left(\frac{1-u}{e^t-u} \right) \times \left(\frac{1-u}{e^t-u} \right) \times \cdots \times \left(\frac{1-u}{e^t-u} \right)}_{N \text{ - times}} e^{xt} \\
 &= \left(\sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} x^{n-l} H_l^{(N)}(u) \right) \frac{t^n}{n!}.
 \end{aligned}
 \tag{58}$$

By comparing coefficients on both sides of (58), we get

$$H_n^{(N)}(x|u) = \sum_{l=0}^n \binom{n}{l} x^{n-l} H_l^{(N)}(u).
 \tag{59}$$

By the definition of notation, we get

$$\begin{aligned} F^{(k)}(t, x) &= F^{(k)}(t)e^{tx} = \left(\sum_{l=0}^{\infty} H_{l+k}(u) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} H_{l+k}(u) x^{n-l} \right) \frac{t^n}{n!}. \end{aligned}$$

From (6), we note that

$$\begin{aligned} (60) \quad \sum_{n=0}^{\infty} H_n^{(N)}(x|u) \frac{t^n}{n!} &= \underbrace{\left(\frac{1-u}{e^t-u} \right) \times \cdots \times \left(\frac{1-u}{e^t-u} \right)}_{N \text{ - times}} e^{xt} \\ &= \left(\sum_{l_1=0}^{\infty} H_{l_1}(u) \frac{H_{l_1}(u)}{l_1!} t^{l_1} \right) \times \cdots \times \left(\sum_{l_N=0}^{\infty} \frac{H_{l_N}(u)}{l_N!} t^{l_N} \right) \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N+m=n} \frac{H_{l_1}(u)H_{l_2}(u)\cdots H_{l_N}(u)}{l_1!l_2!\cdots l_N!m!} x^m n! \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l_1+\cdots+l_N+m=n} \binom{n}{l_1, \dots, l_N, m} H_{l_1}(u) \cdots H_{l_N}(u) x^m \right) \frac{t^n}{n!}. \end{aligned}$$

By comparing coefficients on both sides of (58), we get

$$H_n^{(N)}(x|u) = \sum_{l_1+\cdots+l_N+m=n} \binom{n}{l_1, \dots, l_N, m} H_{l_1}(u) \cdots H_{l_N}(u) x^m.$$

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